OPTIMAL TRANSFERS BETWEEN ELLIPTIC COPLANAR ORBITS (TIME OPEN)

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SUMMARY

From the viewpoint of characteristic speed, the optimal transfers between elliptic coplanar orbits are of 4 types: by one, two, or three finite impulses, or else "through infinity" (theoretically utilizing intermediate orbits which are parabolic, or practically, very elongated ellipses). Of course, each impulse can be arbitrarily subdivided since the transfer time is not limited. Thus, a rocket incapable of furnishing the impulses can nevertheless approach as near as one wishes to the optimum (in a sufficiently long time). The transfers by one or three impulses are very rare. The latter only appear for large eccentricities: it is necessary for the sum of the eccentricities of the initial and final orbits to exceed 1.712. When the optimal transfer is not "through infinity," the search for the optimal impulses (and therefore for the one or two intermediate ellipses) is generally very difficult. It is only easy in the following cases (only the first case is shown here): transfers between: coaxial orbits (perigees on the same side); inverted coaxial orbits (perigees on opposite sides); equal orbits; quasi-circular, neighboring orbits; quasi-circular, not neighboring orbits; orbits of which one has an eccentricity near one.

The third part gives the method of determining the optimal number of impulses (by one, two, or three impulses or else "through infinity") in the majority of cases.

INTRODUCTION

The problem of orbital transfer and that of spatial rendezvous (to rejoin--in position and speed--a real or fictitious moving point circulating in space) are obviously problems essential to space exploration.

One of the severe constraints in these problems is the limited performance of space propulsion systems. It is therefore very interesting to search for the most economical way to effect a transfer or rendezvous.

The problem thus posed is very complex, even in the case of Keplerian orbits removed from every perturbing influence, i.e., a Newtonian field of attraction where there is only one center of mass and it is fixed in position and time; one is scarcely able to resolve the following cases:

1. transfer or rendezvous on an orbit infinitely near to the orbit of departure when time is limited and fixed;
2. transfer or rendezvous on any orbit without limitation of duration.

It is the latter case which is studied here (also being restricted to the study of coplanar orbits). The economy of propellant mass always corresponds to the economy of characteristic speed, whether speed of ejection is modulated or not (the optimization always leads to the use of the maximal speed of ejection). Therefore the problem is always to find the transfer of minimal characteristic speed in each case.

It happens often that one obtains (some) transfers of prohibitive time, and even of infinite duration. The study will give then a lower bound for fuel consumption which one would be able to approach in a sufficiently long time.

Some similar problems have already been dealt with (see references), most often with restrictive hypotheses (limited number of impulses, etc....). There are also numerous numerical studies.

1. FIRST PART

1.1 Definition of the Optimal Character of the Transfer

There is only one center of attraction of a given mass.

At the time $t_1 = 0$ a moving point is on a given elliptic or parabolic Keplerian orbit. One wishes that at the time $t_2 = +\infty$ it would be on another prespecified orbit, that the minimum characteristic speed be expended in the transfer (and therefore the minimum fuel consumption).

The problem of the rendezvous: to rejoin a real or fictional moving point circulating on the final orbit. With time open, this maneuver is no more expensive than the simple transfer because one can always, in order to achieve the transfer, wait for "the good moment" on an orbit neighboring the final orbit and of a slightly different period.

Therefore we have only problems of simple transfers.

1.2 Notations

The direction of the perigee of the first orbit is taken for the direction of reference, with positive rotation in the direction of motion.

The first orbit is entirely defined by a_1 --its semimajor axis and e_1 --its eccentricity, the second is given by a_2, e_2 and ϖ_2 (longitude of the perigee).

We will use the regular notations and reserve the non-subscripted letters for the "actual" or "osculatory" orbit.

$\vec{\gamma}$ = acceleration due to the action of the propellants.

$V_c = \int_0^t |\vec{\gamma}|\, dt$ = characteristic speed.

a = semimajor axis.

e = eccentricity.

$b = a\sqrt{1 - e^2}$ = semiminor axis.

$p = a(1 - e^2)$ = semi latus rectum.

$c = ae$ = focal distance.

ϖ = longitude of the perigee.

v = true anomaly.

E = eccentric anomaly.

n = mean motion.

$\mu = n^2 a^3$ = gravitational constant.

$\vec{r}$ = radius vector.

$\vec{V}$ = speed vector.

$\vec{H} = \vec{r} \times \vec{V}$ = angular momentum vector.

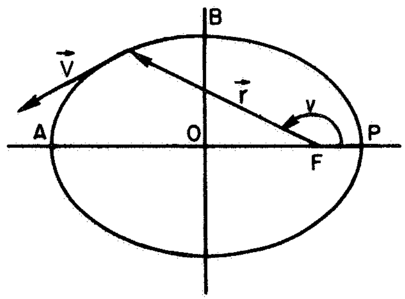
$H = |\vec{H}| = nab$ = length of the angular momentum vector.

$\xi = -\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{V^2}{2}$ = energy.

$P = a(1 - e)$ = distance to the perigee. $0 \le P \le p \le b$

$A = a(1 + e)$ = distance to the apogee. $b \le a \le A$

Fig. 1. Orbit.



OA = OP = FB = a
OB = b
OF = c
FP = P
FA = A

$$r = \frac{p}{1 + e \cos v} = a(1 - e \cos E)$$

1.3 Action of the Engine Thrust

An acceleration $\vec{\gamma}$ caused by propulsive action will be decomposed into:

$S = \gamma \sin \phi$ = radial acceleration (positive in the upward direction)
$T = \gamma \cos \phi$ = horizontal acceleration (positive in the forward direction)

ϕ is oriented in an opposite sense from v and $\bar{\omega}$.

One obtains the classical formulas:

$$\frac{da}{dt} = \frac{2a}{nb}\left[S\ e \sin v + T(1 + e \cos v)\right]$$

$$\frac{de}{dt} = \frac{b}{na^2}\left[S \sin v + T(\cos v + \cos E)\right]$$

$$\frac{d\bar{\omega}}{dt} = \frac{b}{nac}\left[-S \cos v + T\left(\sin v + \frac{a}{b} \sin E\right)\right]$$

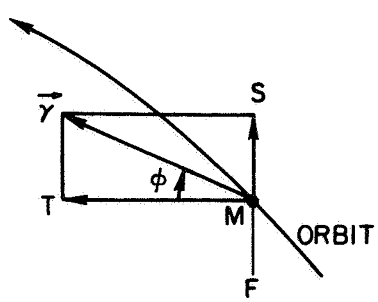
One can deduce from this:

$$\frac{dH}{dt} = rT; \qquad \frac{dp}{dt} = \frac{2b}{na^2} rT \qquad (H^2 = \mu p)$$

$$\frac{d\xi}{dt} = \frac{\mu}{H}\left[S\ e \sin v + T(1 + e \cos v)\right]$$

$$\frac{db}{dt} = \frac{1}{n}\left[T\left[2 + e(\cos v - \cos E)\right] + S\ e \sin v\right]$$

Fig. 2. Thrust.



Remark: The transfer between one parabola and another demands a negligible characteristic speed: an infinitely small impulse at infinite distance is sufficient for passing to a circular orbit, and another one for passing to the second parabola. Therefore, the optimal transfer between two elliptic orbits never utilizes greater than parabolic speeds.

2. SECOND PART

2.1 The Domain of Maneuverability

It is easy enough to demonstrate that the optimal transfer between two coplanar ellipses is always entirely in the plane of the two ellipses; we will assume this point.

An orbit is defined by a, e, and $\bar{\omega}$. The state of a rocket can now be defined by a, e, $\bar{\omega}$ and V_c $(= \int_0^t \gamma dt)$ or by b, e, $\bar{\omega}$, V_c.

One can change independent variables and thereby eliminate time, the maneuverability being canonical and a function only of the parameters b, e, $\bar{\omega}$, and V_c; one obtains then in order to study the actions of an infinitely small thrust $\gamma = dV_c$:

$$\frac{db}{dV_c} = \frac{1}{n}\left[\cos\phi\,[2 + e(\cos v - \cos E)] + e \sin v \sin\phi\right]$$

$$\frac{de}{dV_c} = \frac{b}{na^2}\left[\cos(v - \phi) + \cos\phi \cos E\right]$$

$$\frac{d\bar{\omega}}{dV} = \frac{b}{nac}\left[\sin(v - \phi) + \frac{a}{b}\sin E \cos\phi\right]$$

$$\frac{dV_c}{dV_c} = 1$$

The control parameters are v (attached to E) and ϕ. One can in effect always choose ϕ, direction of the thrust, and v, point of the orbit where thrust is applied. It is always possible to wait for the best position along the orbit (this is permissible when time is open).

Let us suppose, in order to simplify: $\frac{na^2}{b}\frac{de}{dV_c} = X$, $\frac{ncd\bar{\omega}}{dV_c} = Y$, $\frac{ndb}{dV_c} = Z$

The domain of maneuverability is described by:

$$X = \cos(v - \phi) + \cos\phi \cos E = \frac{na^2}{b}\frac{de}{dV_c}$$

$$Y = \frac{b}{a}\sin(v - \phi) + \sin E \cos\phi = \frac{ncd\bar{\omega}}{dV_c}$$

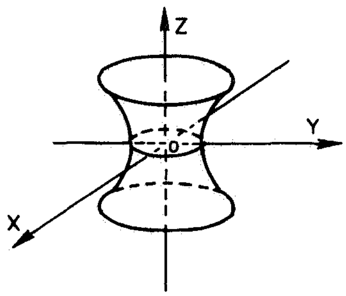
$$Z = 2\cos\phi + e\cos(v - \phi) - e\cos E\cos\phi = n\frac{db}{dV_c}$$

Hence it depends only upon e because $\frac{b}{a} = \sqrt{1 - e^2}$

The changing of ϕ into $\phi + \pi$ changes X into -X, Y into -Y, Z into -Z; the point O (X = 0; Y = 0; Z = 0) is therefore the center of symmetry of the domain.

Likewise the plane OXZ and the axis OY (v into -v and ϕ into $-\phi$) are elements of symmetry of the domain.

Fig. 3. Domain of maneuverability for e = 0.



$|X|$ and $|Z|$ maximum are worth +2 and are obtained for $\phi = 0$ or π, and v = E = 0 or π. The domain of maneuverability then comprises the square +2, -2 in the plane OXZ.

It is easy to verify that $4 \geq X^2 + Y^2 \geq Z^2$; the domain is therefore always at the interior of the cylinder defined by $X^2 + Y^2 \leq 4$, $|Z| \leq 2$.

Consider e = 0 ($\frac{b}{a} = 1$; v = E)

$$\left.\begin{aligned} X &= 2 \cos\phi \cos v + \sin\phi \sin v \\ Y &= 2 \sin v \cos\phi - \sin\phi \cos v \\ Z &= 2 \cos\phi \end{aligned}\right\} X^2 + Y^2 = 1 + \frac{3}{4} Z^2$$

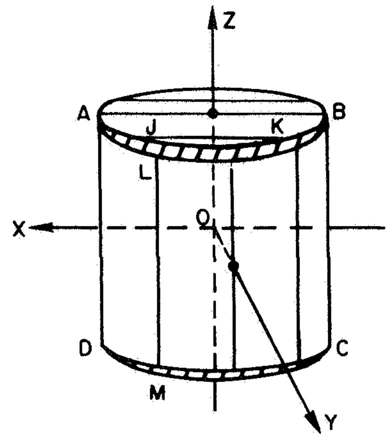
The domain of maneuverability for e = 0 is formed of a hyperboloid of one sheet (Fig. 3) of axis OZ limited to $Z = \pm 2$. Completing the domain by the smallest convex volume which contains it, one obtains for the total domain, the cylinder defined above ($X^2 + Y^2 \leq 4$; $|Z| \leq 2$); the only useful points of the domain are those situated on the surface of this cylinder--that is, those for which $Z = \pm 2$, with either $\phi = 0$ or π. Any other value for ϕ is not optimal.

<u>Study for e ~ 0</u>

The domain of total maneuverability has a slightly different aspect as represented in Fig. 4.

It contains the square ABCD (+2; -2) in the plane OXZ, and comprises 2 convex zones (shaded) corresponding to the useful values of ϕ (which are about from $\frac{e \sin v}{3}$ to $\frac{e \sin v}{2}$ and from $\pi + \frac{e \sin v}{3}$ to $\pi + \frac{e \sin v}{2}$). These zones are pinched at A, B, C and D. The domain comprises, in addition, three developable surfaces: one lateral with such generatrices as LM ($\sim //OZ$), the two others above and below with such generatrices JK ($\sim //OX$). The generatrices correspond to some possibilities of commutation (switching), that is, to discontinuities in the position of the thrust application point.

Fig. 4. Domain of maneuverability for $e \sim 0$.



<u>Study for $e \simeq 1$</u>

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \text{ ; then two cases:}$$

$$\left.\begin{aligned} X &= \cos\phi + \cos(v - \phi) \\ Y &= 0 \\ Z &= X \end{aligned}\right\} \quad \begin{aligned} &v \text{ arbitrary} \\ &E \sim 0 \end{aligned}$$

$$\left.\begin{aligned} X &= \cos\phi(\cos E - 1) \\ Y &= \cos\phi \sin E \\ Z &= -X \end{aligned}\right\} \quad \begin{aligned} &v \sim \pi \\ &E \text{ arbitrary} \end{aligned}$$

One easily deduces from this the domain of total maneuverability (Fig. 5).

The total domain always contains the square ABCD. It is limited by 4 triangles such as AEF, and 4 cones with vertices A and C being supported on the ellipse:

$$X = -Z = \cos E - 1; \quad Y = \sin E$$

and its counterpart which is symmetric with respect to O.

F(-1, +1, +1) corresponds to $E = \pi/2$ and $\phi = 0$.

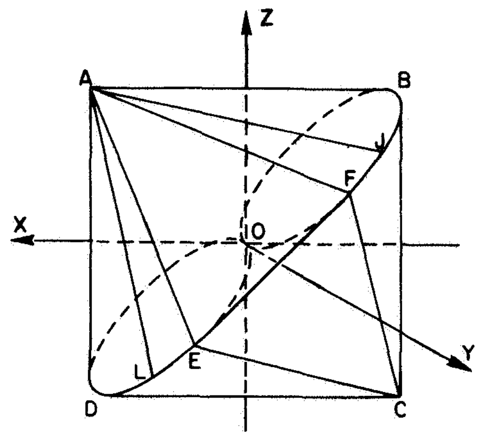
The useful points of the domain are those which are on the surface, that is:

1. A and C for which $v = 0$ and $\phi = 0$ or π;
2. half of the points of the 2 ellipses, for which $\cos E \leq 0$ and $\phi = 0$ or π.

One sees then that for $0 < |E| < \pi/2$ there exist 2 purely ballistic arcs without any useful direction.

The generatrices like AJ or AL represent possibilities of commutation. The triangles like AEF are possibilities of double commutation, that is to say, of infinitesimal optimal transfers by 3 impulses.

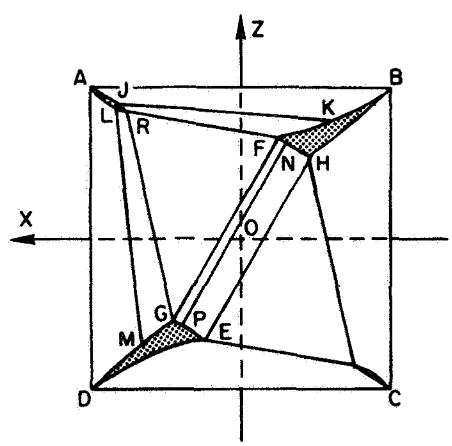
Fig. 5. Domain of maneuverability for $e \sim 1$.



If one studies the same domain with more precision one obtains a slightly different image presented in projection on the plane OXZ in Fig. 6.

There are 4 convex zones (shaded) corresponding to useful values of ϕ. These are pinched respectively at A, B, C and D. There are 4 triangles like RFG, and 6 developable surfaces of which JK, LM, PN are generatrices. (These always correspond to possibilities of commutation.)

Fig. 6. Domain of maneuverability for $e \sim 1$.



General Case

The domain of maneuverability always has the symmetries (O, OY, OXZ). It always lies within the cylinder $X^2 + Y^2 \leq 4$, $|Z| \leq 2$, of which it contains the square ABCD in the plane OXZ. It changes form in a continuous manner from e = 0 until e = 1 and there is a limiting value, e_0, close to 0.925, for which appear triangles which characterize the domain for $e \sim 1$.

$e \leq e_0$: there are two opposed angles by the vertex of useful ϕ values (that is, which can be optimal for a well chosen transfer) for each value of v.

$e > e_0$: certain positions of orbit (in the lower half) have no useful angle. They correspond to necessarily ballistic arcs. For these eccentricities there exist infinitesimal optimal transfers by three impulses.

One precise numerical study has led to $0.9248 < e_0 < 0.9252$. For eccentricity e_0 the domain of maneuverability contains four infinitely flat symmetric triangles, having two peculiar vertices corresponding to $v = \pm 122.31^\circ$; ($E = \pm 39.44^\circ$) and $\phi = 26.04^\circ$ (+0 or 180°). These points are utilized in the course of describing "Lawden's spiral" (1) arc for which

$$e \sin v = \frac{6\, s \sqrt{1 - s^2}\, (1 - 2s^2)(3 - 4s^2)}{(3 - 5s^2)^2} \quad \text{and}$$

$$(1 + e \cos v) = \frac{(1 - 3s^2)(3 - 4s^2)^2}{(3 - 5s^2)^2} \quad \text{with } s = \sin\phi$$

The third vertex of these infinitely flat triangles corresponds to $\phi_1 = \pm 170.33^\circ$ ($+0^\circ$ or 180°), $E_1 = 96.26^\circ$, with the relations: $\sin\phi \sin\phi_1 > 0$; $\cos\phi \cos\phi_1 < 0$; $\sin E \sin E_1 < 0$ and $\sin E \tan\phi > 0$.

2.2 The Useful Angle and the Commutations

The determination of the useful angles is very important, not only because they delimit the utilizable directions of thrust, but also because the extreme directions on the sides of the useful angle correspond to the commutations, that is to say, to the discontinuities in the position of the thrust application point (thrust-off and then on again at another point of the orbit).

Let us rewrite the equations of the domain of maneuverability:

$$X = \cos(v - \phi) + \cos\phi \cos E$$

$$Y = \frac{b}{a} \sin(v - \phi) + \cos\phi \sin E$$

$$Z = 2 \cos\phi + e \cos(v - \phi) - e \cos\phi \cos E$$

At a point XYZ of the domain of maneuverability the tangent plane is determined by the two vectors:

$$\left(\frac{\partial X}{\partial \phi}, \frac{\partial Y}{\partial \phi}, \frac{\partial Z}{\partial \phi}\right)$$

and $$\left(\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}\right)$$

(E being bound to v by $\tan \frac{E}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{v}{2}$).

If the point XYZ corresponds to a limiting direction of the useful angle, it is such that there is a second point X, Y, Z where the tangent plane to the domain is the same (corresponding point of commutation).

It is sufficient then, theoretically, to solve for v and e given ϕ, v_1, and ϕ_1 in the three following equations:

$$\begin{vmatrix} X \; Y \; Z \\ \frac{\partial}{\partial \phi}(X, Y, Z) \\ \frac{\partial}{\partial v}(X, Y, Z) \end{vmatrix} = \begin{vmatrix} X_1 \, Y_1 \, Z_1 \\ \frac{\partial}{\partial \phi}(X, Y, Z) \\ \frac{\partial}{\partial v}(X, Y, Z) \end{vmatrix}; \quad \begin{vmatrix} \frac{\partial}{\partial \phi_1}(X_1, Y_1, Z_1) \\ \frac{\partial}{\partial \phi}(X, Y, Z) \\ \frac{\partial}{\partial v}(X, Y, Z) \end{vmatrix} = 0; \text{ and}$$

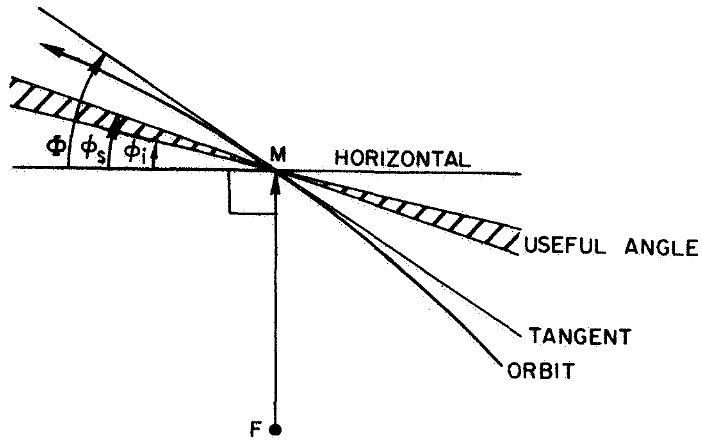
$$\begin{vmatrix} \frac{\partial}{\partial v_1}(X_1, Y_1, Z_1) \\ \frac{\partial}{\partial \phi}(X, Y, Z) \\ \frac{\partial}{\partial v}(X, Y, Z) \end{vmatrix} = 0$$

One can thus obtain the useful local angle and the corresponding commutations for every value of e and v.

We will go on to give here some limited developments in the vicinity of e = 0 and of e = 1.

Because of the symmetries, the useful angle is formed from 2 angles opposed to the vertex. These angles are always within the acute angles formed by the tangent and the local horizontal. The useful angle will be delimited by ϕ_i and ϕ_s (Fig. 7).

Fig. 7. Useful Angle



$\sin v \geq 0$: $0 \leq \phi_i \leq \phi_s \leq \Phi \leq \frac{\pi}{2}$

$\sin v \leq 0$: $0 \geq \phi_i \geq \phi_s \geq \Phi \geq -\frac{\pi}{2}$

1. Development in the vicinity of $e = 1$.

Let us put: $\frac{b}{a} = \sqrt{1 - e^2} = \epsilon$

A. In the vicinity of the perigee ($v \sim 0$), to order ϵ^5:

$$\phi_i = \frac{v}{4} + \frac{v^3}{512} - \frac{3v\epsilon^2}{32}; \quad \phi_s = \frac{v}{4} - \frac{v^3}{512} - \frac{v\epsilon^2}{32};$$

and $|v| \leq |4|\epsilon$ (beyond $|v| = 4\epsilon$: $|\phi_i| > |\phi_s|$, there is no longer a useful angle).

B. In the higher half of the orbit ($v \sim \pi$) to order ϵ^3:

$$\phi_i = \epsilon \sin E \cdot \max \left[\left(\frac{1}{2(1 - \cos E)} \right); \left(-1 + \sqrt{3 + \frac{\cos E}{\epsilon^2}} \right) \right]$$

Let us put $\phi_i = \frac{\epsilon \sin E}{2(1 - \cos E)} = \frac{\sin v}{2}$

And $\phi'_i = \epsilon \sin E \left[-1 + \sqrt{3 + \frac{\cos E}{2}} \right]$; $\phi_s = \frac{3}{2} \sin v = 3\phi_i$

For $\cos E < -\frac{3}{4}\epsilon^2$, $|\phi'_i| < |\phi_i|$; ϕ'_i is not involved. The useful angle goes from ϕ_i to ϕ_s.

For $-\frac{3}{4}\epsilon^2 < \cos E < +\frac{13}{4}\epsilon^2$: $|\phi'_i| > |\phi_i|$; ϕ_i is not involved. The useful angle goes from ϕ'_i to ϕ_s.

For $\cos E > \frac{13}{4}\epsilon^2$: $|\phi'_i| > |\phi_s|$, there is no longer a useful angle.

Thus, from $v = 4\epsilon$ to $E = \frac{\pi}{2} - \frac{13}{4}\epsilon^2$ and from $v = -4\epsilon$ to $E = -\frac{\pi}{2} + \frac{13}{4}\epsilon^2$ are extended to purely ballistic arcs without any useful direction. These arcs occur between $e = 1$ and $e = e_0$ (~ 0.925).

2. Development in the neighborhood of e = 0 or of sin v = 0:

$$\tan\phi_s = \frac{e\sin v(2-e\cos v)}{4+e\cos v-e^2\cos^2 v} - \frac{2e^3\sin^3 v(6-e^2\cos^2 v)^2}{(3+e\cos v)(4+e\cos v-e^2\cos^2 v)^4} - \frac{13}{1536}e^5\sin^5 v + \text{order}\,(e^6\sin^5 v)$$

$$\tan\phi_i = \frac{e\sin v}{3+e\cos v} + \frac{18e^3\sin^3 v}{(3+e\cos v)^5} + \frac{2}{81}e^5\sin^5 v + \text{order}\,(e^6\sin^5 v)$$

(true for ϕ_i but not for ϕ_i')

For the sake of comparison: $\tan\Phi = \dfrac{e\sin v}{1+e\cos v} = \dfrac{e\sin E}{\sqrt{1-e^2}}$

$$\left.\begin{array}{ll} \phi_s & \text{maximum} \sim 26.2^\circ \\ (\phi_s-\phi_i) & \text{maximum} \sim 12^\circ \\ \dfrac{(\phi_s-\phi_i)}{\Phi} & \text{maximum} < 0.2 \end{array}\right\}$$

The useful angle is always less than 12.5° and less than one fifth of Φ (angle between the tangent and the local horizontal).

One always has: $\left|\dfrac{e\sin v}{3+e\cos v}\right| \leq |\tan\phi_i|$;

$$|\tan\phi_s| \leq \left|\frac{e\sin v(2-e\cos v)}{4+e\cos v-e^2\cos^2 v}\right|$$

Now here are the corresponding commutations: v, E, ϕ, ϕ_s, ϕ_i, ϕ'_i are relative to the point of arrival on the intermediary ellipse of the commutation. v_1, E_1, ϕ_1, ϕ_{s1}, ϕ_{i1}, ϕ'_{i1} are relative to the point of departure of the intermediary ellipse (corresponding point of commutation). Of course all these elements are related to the intermediary ellipse.

1. Commutations relative to ϕ_s.
(They correspond to the generatrices JK in the sketches of the domain of maneuverability.)

$$\tan\frac{v}{2}\cdot\tan\frac{v_1}{2} = \frac{(3+e)(4+e-e^2)}{(3-e)(4-e-e^2)} + (\text{order } e^3\sin^2 v)$$

The calculation of v_1 by this formula gives an error of the order of $e^3\sin^3 v$.

Here is a more precise expression:

$$\tan\frac{v}{2}\cdot\tan\frac{v_1}{2} = \frac{(3+e)(4+e-e^2)}{(3-e)(4-e-e^2)}\left[1+\alpha\cos^2\left(\frac{v-v_1}{2}\right)\right] + \left(\text{order } e^5\sin^2 v\sin^2\frac{v-v_1}{2}\right)$$

obtained in utilizing Lawden's results on an arc with intermediate thrust (for this arc: $v = v_1$ $(= v_0)$, α is therefore determined by:

$$\frac{3+e}{3-e}\cdot\frac{4+e-e^2}{4-e-e^2}\cdot(1+\alpha) = \frac{1-\cos v_0}{1+\cos v_0} = \tan^2\frac{v_0}{2}$$

v_0 being determined by:

$$e \sin v_0 = \frac{6s\sqrt{1-s^2}(1-2s^2)(3-4s^2)}{(3-5s^2)^2}$$

$$e \cos v_0 = \frac{3s^2(-7+21s^2-16s^4)}{(3-5s^2)^2}$$

s (which is used in place of $\sin \phi$) serves as an auxiliary parameter.

One obtains:

$$v_0 = \frac{\pi}{2} + \frac{7e}{12} + \frac{301e^3}{10.368} + \text{(order } e^5)$$

$$a = \left[-\frac{31}{864}e^3 + \text{(order } e^5)\right]$$

The calculation of v_1 by this last expression gives an error of the order of $e^5 \sin^3 v \sin^2 ((v - v_1)/2)$

In particular: for $e \sim 0$: $v_1 \sim \pi - v$
for $e \sim 1$: $\sin v \sim 0$ (because of the two ballistic arcs) and $\tan \frac{v}{2} \cdot \tan \frac{v_1}{2} \sim 4.$

ϕ and ϕ_1 are given by:

$$\phi - \phi_s = \phi_1 - \phi_{s1} = \begin{cases} 0 \text{ if } \cos v > \cos v_1 \\ \pi \text{ if } \cos v < \cos v_1 \end{cases}$$

2. Commutations relative to ϕ_i:
(They correspond to the generatrices LM.)

$$\tan\frac{v}{2} \cdot \tan\frac{v_1}{2} = -\frac{(3+e)^2}{(3-e)^2} + \text{(order } e^3 \sin^2 v)$$

which gives an error of the order of $e^3 \sin^3 v$ in the calculation of v_1. Here is a more precise expression:

$$\cot\frac{v}{2} \cdot \cot\frac{v_1}{2} = -\frac{(3-e)^2}{(3+e)^2}\left[1 + \beta \cos^2\left(\frac{v+v_1}{2}\right)\right] + \left[\text{order } e^5 \sin^2 v \sin^2\left(\frac{v+v_1}{2}\right)\right]$$

β being defined by: $(1+\beta)\dfrac{(3-e)^2}{(3+e)^2} = \dfrac{1+x}{1-x}$

with x = root of $e^2x^3 + 3ex^2 + (3+e^2)x + 2e = 0$

$$\left(\beta = -\frac{16e^3}{27} - \frac{40e^5}{81} + \frac{128e^6}{729} + \dots\right)$$

The calculation of v_1 by this last expression gives an error of the order of $e^5 \sin^3 v \sin^2\left(\frac{v + v_1}{2}\right)$.

In particular: for $e = 0$: $v_1 = \pi + v$
for $e \sim 1$: $\tan(v/2) \cdot \tan(v_1/2) \sim -4$

ϕ and ϕ_1 are given by $\phi = \phi_i$; $\phi_1 = \phi_{i1} + \pi$.

3. Commutations relative to ϕ_i' (therefore for $e > e_0 \sim 0.925$).
(These commutations correspond to the generatrices PN in the last sketch of the domain of maneuverability, for which $e \sim 1$.)

Let us put $\epsilon = \sqrt{1 - e^2}$ and perform the development in the vicinity of $\epsilon = 0$.

The parameter $z(|z| \le \frac{\epsilon}{2})$ is useful in describing the solutions (with an error of the order of ϵ^3 for ϕ_i' and ϕ_{i1}' and ϵ^4 for E and E_1):

$$|z| \le \frac{\epsilon}{2}, \quad \phi_i' = z \pm \epsilon, \quad \phi_{i1}' = z \mp \epsilon, \quad (\phi_i' \phi_{i1}' \le 0)$$

$$E = -4\epsilon z + \operatorname{sgn} \phi_i' \left(\frac{\pi}{2} - \epsilon^2 - z^2\right)$$

$$E_1 = -4\epsilon z + \operatorname{sgn} \phi_{i1}' \left(\frac{\pi}{2} - \epsilon^2 - z^2\right)$$

$$\phi = \phi_i', \quad \phi_1 = \phi_{i1}' + \pi$$

If one terms an accelerating thrust a thrust directed into the part before the useful angle (see Fig. 7), and braking a thrust directed into the back part, one sees that: the sense of the commutations is such that, one commutation ϕ_i or ϕ_i' follows an accelerating thrust and precedes a brake; one commutation ϕ_s follows a low* accelerating thrust and precedes a high acceleration, or else follows a high braking thrust and precedes a lower one. This determines the sense of the commutations, which we will demonstrate further.

For the commutations ϕ_i (or ϕ_i' if $e > e_0 \sim 0.925$) the case $v_1 = -v$ is simple. One obtains there:

$\cos v = \cos v_1$ = the root of the following equation:

$$e^2x^3 + 3ex^2 + (3 + e^2)x + 2e = 0$$

from which $\cos E = \cos E_1$ = the root of the following equation:

$$e^2y^3 + y^2(e^3 - 3e) + y(3 - 2e^2) = e(1 - e^2)$$

* refers to a low <u>position</u> on the orbit

$$\phi = \phi_i (\text{or } \phi'_i) = \text{arc tan} \frac{e \sin v}{3 + 2e \cos v} = \text{arc tan} \frac{\cos E}{\sin v}$$

$$\phi_I = \phi_{iI} + \pi \ (\text{or } \phi'_{iI} + \pi) = \pi - \phi$$

This case is utilized in the transfer between equal ellipses.

2.3 Impulses and Continuous Thrusts

At this point, we can see that an optimal transfer between coplanar orbits can be presented in the following manner: it is composed of a series of continuous thrusts or of "retrograde thrusts" (where the point of optimal application of the thrust is continuously displaced in a direction opposite to the primary motion, and for which a physical interpretation consists of an infinity of on and off switches of the thruster with coasts of nearly an orbital revolution between each thrust), or even of impulses (the optimal point of application is fixed). These continuous or "retrograde" thrusts, or else these impulses, are:

1. always applied within the useful local angle.
2. separated from one another by the commutations that we discovered in the preceding chapter, and which correspond to the discontinuities in position from the optimal application point of the thrust.

In fact, there are never (optimal) continuous thrusts or optimal retrograde thrusts. One can see this in the following manner.

Let us see under what conditions a determined impulse is "locally optimal," i.e., cannot be replaced by 2 or more impulses at infinitely neighboring points (or in the limit by a continuous or "retrograde" thrust) Fig. 8.

Fig. 8. Study of local optimality.

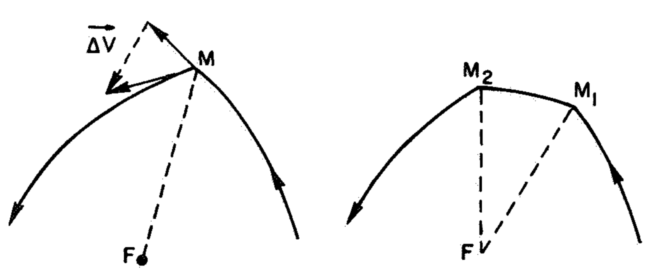
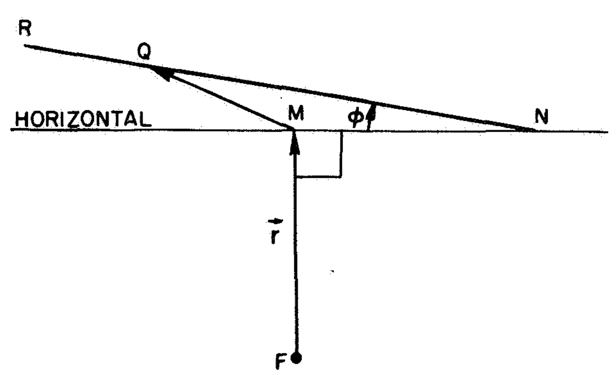


Fig. 9. Local optimality.



The calculation is simple but long and ought to be carried out to the second order.

The result is (Fig. 9):

1. The character of local optimality depends only on the point of application and on the direction of the impulse, <u>and not on its size</u>.

2. If one poses: $\overrightarrow{MQ} = \overrightarrow{V}$ = initial speed of M.
$\overrightarrow{QR} = \overrightarrow{\Delta V}$ = considered impulse.
N at the intersection of QR and of the local horizontal.

If $|MN| \cdot \sqrt{1 - 3\sin^2\phi}$ > local circular speed $= \sqrt{\frac{\mu}{r}}$:

the impulse is locally optimal.

If $|MN| \cdot \sqrt{1 - 3\sin^2\phi}$ < local circular speed $= \sqrt{\frac{\mu}{r}}$:

the impulse is not locally optimal.

The character of local optimality does not depend on the size of the $\overrightarrow{\Delta V}$'s and, the globally optimal thrusts being also locally optimal, it is not possible to have any continuous or "retrograde" thrust for which:

$$|MN| \cdot \sqrt{1 - 3\sin^2\phi} > \text{local circular speed}$$

because a small arc of such thrust would be advantageously replaced by the corresponding impulse.

On the other hand, this demonstration would not be valid if on an arc of continuous or retrograde thrust one had constantly $|MN| \cdot \sqrt{1 - 3\sin^2\phi}$ = local circular speed, with the additional condition of course that the direction of thrust is within the useful angle. (The direction in question could at most be a limiting direction among those globally optimal, i.e., one will have $\phi = \phi_i$ or ϕ_s (+0 or π).) One finds effectively that $|MN| \cdot \sqrt{1 - 3\sin^2\phi}$ = local circular speed for $\phi = \phi_s$ when the corresponding commutation ϕ_s is infinitely short, i.e., when:

$$v = v_1 = \pm \left(\frac{\pi}{2} + \frac{7e}{12} + \frac{301e^3}{10.368} + (\text{order } e^5) \right)$$

This is quite logical because an infinitely short commutation is "local" and thrusts in its immediate vicinity are not locally optimal.

Thus appears an arc of continuous thrust (or, by symmetry, retrograde thrust) which is the limiting case for a great number of short commutations ($v \sim v_1$). This arc is that which is described by Lawden (1). Nevertheless it is not optimal because the optimal number of commutations remains limited. But it is far from being large: if, in departing from an ellipse of eccentricity e, one follows such an arc with central angle α, the relative loss (for small e) in relation to the optimum is only:

$$\frac{e^2}{32}\left(\frac{3\alpha^2}{\tan^2\frac{\alpha}{2}} + 2\alpha^2 - 12 \right) + (\text{order } e^3), \quad \text{if } \left|\tan\frac{\alpha}{2}\right| \geq \frac{\alpha}{2}$$

which is $\frac{e^2\alpha^4}{640}$ for small α, $e^2\,\frac{\pi^2 - 6}{16}$ for $\alpha = \pi$

if $\left|\tan\frac{\alpha}{2}\right| \leq \frac{\alpha}{2}$ the error is

$$\frac{3e^2}{8}\left(1 - \frac{4\sin^2\frac{\alpha}{2}}{\alpha^2}\right)\left(\left|\cos\frac{\alpha}{2}\right| + \sqrt{\frac{\alpha^2}{4} - \sin^2\frac{\alpha}{2}}\right)^2 - \frac{e^2\alpha^2}{32} + (\text{order } e^3)$$

which is $\frac{e^2}{8}(2n^2\pi^2 + 6n\pi + 3)$ for $\alpha = 2n\pi$ (n integer)

2.4 The Senses of the Commutations

An optimal coplanar transfer allows only one impulse (finite or infinitely small) between 2 successive commutations. Since the commutations occur when the direction of thrust is coincident with that from one of the extremes of the useful angle, one can determine the sense of the commutations by studying the sense of variation of these limiting angles in the course of an "impulsive" thrust.

ϕ_i, ϕ_i' and ϕ_s only depend on e and v;

$\frac{de}{dV_c}$ is known, $\frac{dv}{dV_c} = -\frac{d\bar{\omega}}{dV_c}$ because $(v + \bar{\omega})$ is fixed in the course of an impulse.

From which:

1. $\phi \sim \phi_i$ (or ϕ_i') + 0 or π

A) For $e \sim 0$ or $\sin v \sim 0$: $\frac{d\tan\phi_i}{dV_c} = \frac{2be\sin v\cos\phi}{na^2(3+e\cos v)^2(1+e\cos v)} + (\text{order } e^3\sin^3 v)$

B) For $e \sim 1$: $\frac{d\tan\phi_i}{dV_c} = \frac{\sin E\cos\phi}{8na}$ ....if $v \sim 0$, $= \frac{\sin E\cos\phi}{2na}$ ....if $v \sim \pi$

$$\frac{d\tan\phi_i'}{dV_c} = \frac{\cos\phi\sin E}{na}\left(-1 + \frac{3}{\sqrt{3 + \frac{\cos E}{1-e^2}}}\right)$$

let $\frac{1}{2} \leq \frac{na}{\cos\phi\sin E} \cdot \frac{d\tan\phi_i'}{dV_c} \leq \frac{3}{2}$

C) For all cases:

$$\boxed{\operatorname{sgn}\frac{d\phi_i\,(\text{or }\phi_i')}{dV_c} = \operatorname{sgn}\cos\phi\sin v\,; \quad \text{for } \phi \sim \phi_i \text{ (or } \phi_i') + 0 \text{ or } \pi}$$

2. $\phi \sim \phi_s$ or $\phi_s + \pi$

A) For $e \sim 0$ or $\sin v \sim 0$:

$$\frac{d\tan\phi_s}{dV_c} = -\frac{b\cos\phi}{na^2}\cdot\left[\frac{6e^2\sin v\cos v}{(4 + e\cos v - e^2\cos^2 v)^2(1+e\cos v)} + \frac{7e^3\sin^3 v}{32} + \text{order }(e^4\sin^3 v)\right]$$

B) For $e \sim 1$:

$$\frac{d \tan \phi_s}{dV_c} = -\frac{3}{8}\frac{\sin E \cos\phi}{na} \ldots\ldots \text{if } v \sim 0, + \frac{3}{2}\frac{\sin E \cos\phi}{na} \ldots\ldots \text{if } v \sim \pi$$

C) For all cases by setting v_1 = the correspondant of v in the commutation ϕ_s:

$$\boxed{\begin{array}{l} \operatorname{sgn} \dfrac{d\phi_s}{dV_c} = \operatorname{sgn}\left[\cos\phi \cdot \sin v \cdot (\cos v_1 - \cos v)\right]; \\ \text{for } \phi = \phi_s \text{ or } \phi_s + \pi \end{array}}$$

From the two preceding framed relations and from:

$$\left.\begin{array}{l} 0 \le \phi_i (\text{or } \phi_i') \le \phi_s \le \frac{\pi}{6} \text{ for } \sin v \ge 0 \\ 0 \ge \phi_i (\text{or } \phi_i') \ge \phi_s \ge -\frac{\pi}{6} \text{ for } \sin v \le 0 \end{array}\right\} \text{with } \phi \text{ or } \phi + \pi \in \left[\phi_i (\text{or } \phi_i'), \phi_s\right]$$

one obtains the sense of the commutations:

1. Commutations ϕ_i (or ϕ_i'):

$$\phi = \phi_i \ (\text{or } \phi_i'); \quad \phi_1 = \phi_{i1} (\text{or } \phi_{i1}') + \pi$$

The commutations ϕ_i (or ϕ_i') follow an accelerating thrust and precede a brake (situated at a point of the orbit on the other side of the major axis).

2. Commutations ϕ_s:

$$\phi - \phi_s = \phi_1 - \phi_{s1} = \begin{cases} 0 \text{ if } \cos v > \cos v_1 \\ \pi \text{ if } \cos v < \cos v_1 \end{cases}$$

The commutations ϕ_s involve one accelerating thrust followed by another at a higher point in the orbit, or else a brake by another at a lower point (on the same side of the major axis).

It is then easy to construct examples of optimal transfer in departing from an intermediate ellipse, as well as sense of the commutation above. The opposite problem, to find an intermediary ellipse or ellipses to depart from the given orbits, can only be resolved implicitly.

2.5 The Number of Commutations

A "complete" optimal coplanar transfer contains therefore: a series of accelerating thrusts at points higher and higher which, according to the sign of $\bar{\omega}_2$ (for $-\pi \le \bar{\omega}_2 \le +\pi$) will be accomplished in an ascending phase (for $\bar{\omega}_2 > 0$) or descending (for $\bar{\omega}_2 < 0$); then, after a commutation ϕ_i, (or ϕ_i'), a series of brakes at lower and lower points in the other phase.

In fact, there are only the following cases:

A. Finite transfers by one, two or three impulses with 0, 1, or 2 commutations. In the last case, one commutation is ϕ_s and one is ϕ_i (or ϕ'_i).

B. Transfers "through infinity," which are simple although time consuming. A tangential impulse at perigee effects transfer to a parabola. A return parabola is entered by some negligible impulses at a great distance. Passage onto the final ellipse is by a brake which is tangential to the perigee. This transfer mode is always optimal if the ratio of the distances to the perigees is greater than 11.938, or, between non-coplanar orbits, if the angle of the orbital planes is greater than 60.1850°.

This mode of transfer always has a characteristic speed between 41.42-50% of the sum of the two apogee speeds:

$$V_{c\infty} = \sqrt{\frac{2\mu}{P_1}} - \sqrt{\frac{\mu A_1}{a_1 P_1}} + \sqrt{\frac{2\mu}{P_2}} - \sqrt{\frac{\mu A_2}{a_2 P_2}}$$

Obviously, this transfer is not physically realizable. But one can come as close as one wishes, in a sufficiently long time, by using two very elongated ellipses and, at great distance, a large circle or a large intermediate ellipse.

In order to demonstrate propositions A and B above, it is sufficient to prove that a transfer using the series of commutations ϕ_s, ϕ_s; or ϕ_s, ϕ_i, ϕ_s; or ϕ_s, ϕ'_i, ϕ_s cannot be optimal.

It is easy enough to demonstrate that the series ϕ_s, ϕ_i (or ϕ'_i), ϕ_s is not optimal. We already know that the series ϕ_s, ϕ_i (or ϕ'_i) and the series ϕ_i (or ϕ'_i), ϕ_s are realizable only if the intermediate ellipse of the commutation ϕ_s has an eccentricity greater than $(\sqrt{3} - 1)$, and that they are only optimal if this same eccentricity is at least of the order of 0.9. Finally, for eccentricities close to 1, the series ϕ_s, ϕ_i (or ϕ'_i), ϕ_s has a characteristic speed at least double that of the transfer "through infinity" between the intermediate ellipses of the commutations ϕ_s.

The demonstrations of the non-optimality of the series ϕ_s, ϕ_s with 2 commutations of type ϕ_s (therefore of 3 accelerating thrusts or of 3 successive brakings) is much more delicate in the general case, although it is evident for $e \sim 1$ because of the discontinuity due to the ballistic arcs. It is made without too much difficulty in cases where $e(1 - e) \sin v$ is 0 or close to 0 at one time or another in the course of the transfer (v being the true anomaly of the points where one thrusts).

3. THIRD PART: GENERALITIES--DETERMINATION OF THE OPTIMAL MODE OF TRANSFER

Since the domain of maneuverability is independant of $\bar{\omega}$ and has the symmetry $(d\bar{\omega}, -d\bar{\omega})$, the result is:

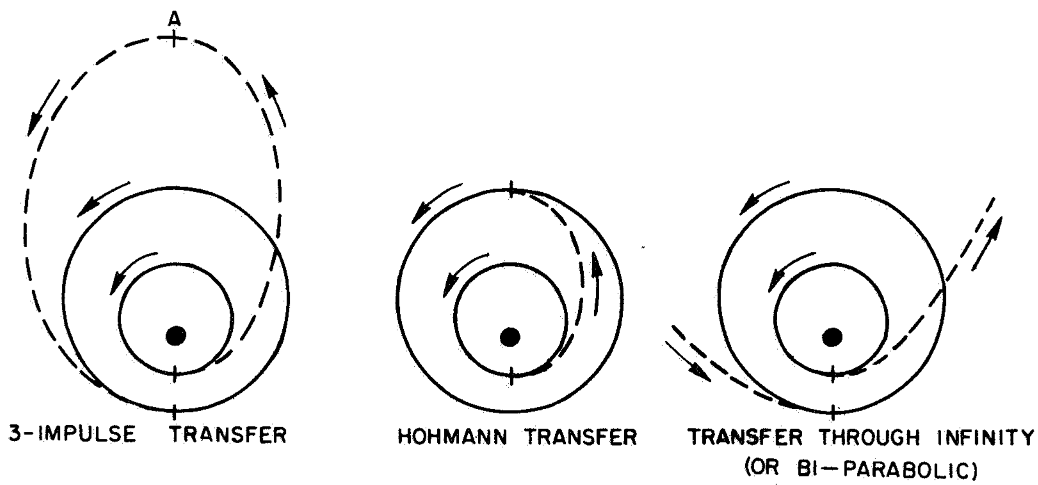
1. The variation of $\bar{\omega}$ is monotonoic in the course of a transfer.

2. The cost of a transfer is an increasing function of $|\bar{\omega}_2|$ (for $-\pi \leq \bar{\omega}_2 \leq + \pi$). In particular, if the optimal transfer is "through infinity" for $\bar{\omega}_2 = 0$, it is the same for all $\bar{\omega}_2$.

3. If $\bar{\omega}_2 = 0$ (aligned coaxial orbits): $\bar{\omega}_2 \equiv 0$. One always uses the points of the domain of maneuverability for which $d\bar{\omega} = 0$, i.e., the points ABC or D (tangential thrusts at the perigee or at the apogee). The series of the impulses was shown in the preceding section. There are then, at most, 3 of them (Fig. 10):

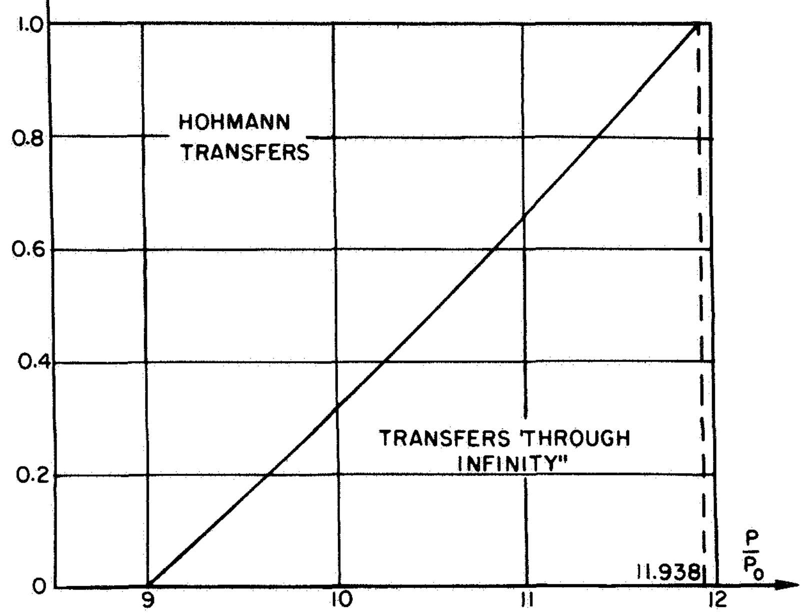
I. A tangential, accelerating impulse at the perigee of the first orbit.
II. A tangential impulse at the apogee A of the intermediate ellipses.
III. A tangential, braking impulse at the perigee of the second orbit.

Fig. 10. Transfers between direct coaxial orbits (or aligned co-axial orbits).



It is easy to compare the remaining possibilities. One sees then that for the optimum, one of the three impulses is infinitely small, and that the transfer is either: 1) "Hohmann's"--by a bi-tangential ellipse, tangent at the higher apogee and at the perigee of the other orbit, or 2) "through infinity" with quasi-parabolas and an infinitely small impulse at a large distance. The occurrence of these two possibilities is illustrated in Fig. 11.

Fig. 11. Transfers between aligned, coaxial, coplanar orbits ($\bar{\omega}_2 = 0$). One sets $A = \max(A_1, A_2)$, $P = \max(P_1, P_2)$, $P_0 = \min(P_1, P_2)$. The optimal transfer is always: "Hohmann's" if $P/P_0 \le 9$, "through infinity" if $P/P_0 \ge 11.938$. $\mathrm{Min}(A_1, A_2)$ never affects the choice.



Now here are some more general remarks:

A. The optimal transfer is by one impulse <u>if and only if</u> the following conditions are met:

1. The two given orbits are tangent or intersecting.
2. At the lowest intersection point (where the optimal impulse is accomplished) the required impulsive ΔV is:

   I. Within the useful local angle of each of the two orbits.
   II. Such that:

$$\Delta V \le V_{c\infty} = \sqrt{\frac{2\mu}{P_1}} - \sqrt{\frac{\mu A_1}{a_1 P_1}} + \sqrt{\frac{2\mu}{P_2}} - \sqrt{\frac{\mu A_2}{a_2 P_2}}$$

These transfers are rare because of condition 2.I. and because of the smallness of the useful angles. This condition requires that if the orbits are tangent they ought to be coaxial ($\bar{\omega}_2 = 0$ or π), which includes the case where one is a circle.

3. The optimal transfers by 3 impulses are also very rare. They are found only if the following (not sufficient) conditions are met:

   1. $\left(\frac{\sqrt{P_1}}{a_1} + \frac{\sqrt{P_2}}{a_2}\right) \cdot \max\left(\sqrt{P_1}, \sqrt{P_2}\right) < \frac{11 - \sqrt{41}}{16} = 0.2873$

   (which carries with it $e_1 + e_2 > 1.712$)

   2. $0^\circ < |\bar{\omega}_2| < 22^\circ$

   3. $\frac{9}{25} < \frac{P_1}{P_2} < \frac{25}{9}$

The conditions 1 and 2 are very restrictive.

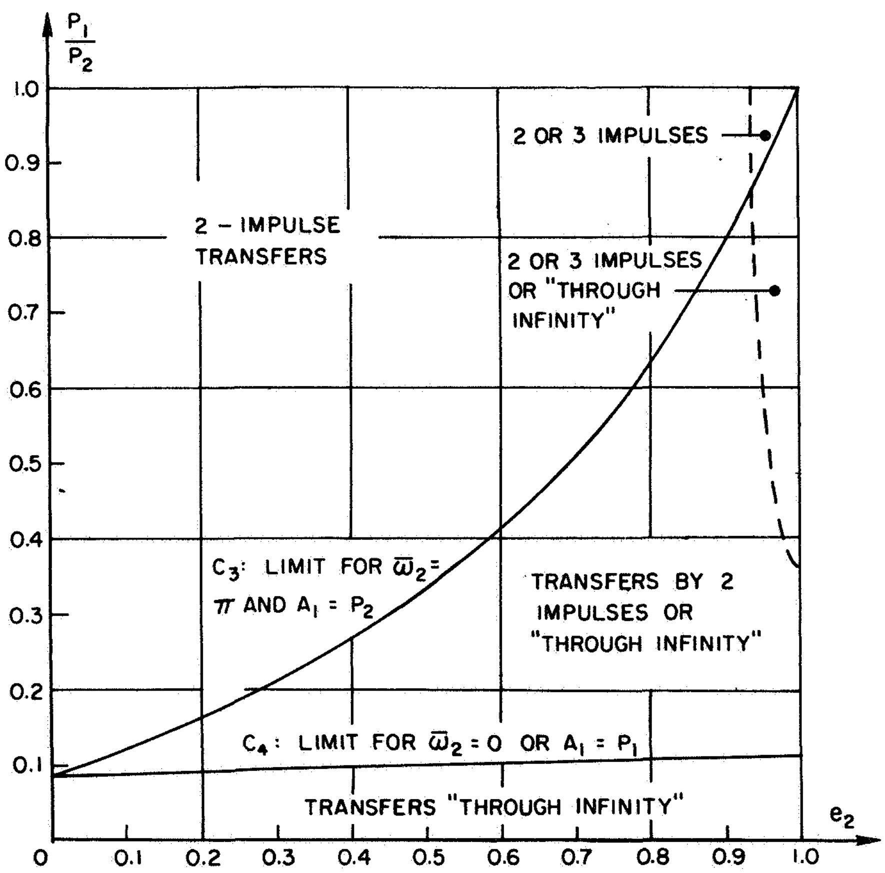
For these transfers, the first impulse is an accelerating thrust, the last a brake.

If the given orbits are tangent and coaxial (aligned or opposing, $\bar{\omega}_2 = 0$ or π), the optimal transfer is either by one impulse at the point of contact or "through infinity." The occurrence of these two possibilities can be seen in Fig. 11 or 12 if $\bar{\omega}_2 = 0$ and Fig. 12 if $\bar{\omega}_2 = \pi$.

In the other cases, the Figures 12 and 13 often permit the determination of the optimal mode of transfer (by 1, 2, or 3 impulses or else "through infinity"). We always suppose here that $P_1 \leq P_2$, which one can always do, since the transfers are reversible.

Fig. 12 describes the case where the orbits do not intersect; e_2 is an abscissa, P_1/P_2 an ordinate.

Fig. 12. Transfers between non-intersecting coplanar orbits (an exception is made in the case where they are tangent and coaxial). $P_1 \leq A_1 < A_2$, $P_1 < P_2 \leq A_2$. If the transfer is "finite" the first two impulses are accelerating thrusts, the third is a brake. If $A_1 + P_1 (= 2a_1) \leq 6.32\ P_2$, the optimal transfer cannot be by three impulses.



The curve C_4 therein is a transformation of the curve in Fig. 11.

The equation of the curve C_3 is (where $P_1/P_2 = y$):

$$\sqrt{2 + 2e_2} = \left(1 + \frac{1}{\sqrt{y}}\right)\left(1 - \frac{1 - \sqrt{y}}{\sqrt{1 + y}}\right)$$

It passes by the point: $e_2 = \frac{1}{2}$; $y = \frac{P_1}{P_2} = \frac{1}{3}$

The dotted curve (upper right), showing the limit of the domain of transfers "by 3 impulses," is traced "by judgment" to pass through the points $e_2 = 1$, $P_1/P_2 = 0.36$ and $e_2 \sim 0.935$ or 0.940; $P_1/P_2 = 1$.

In doubtful cases the following sufficient condition for "passage through infinity" (valuable here but not always) sometines permits the determination:

$$(\sqrt{2} - 1)(1 - \cos \bar{\omega}_2) \geq \min\left(\frac{1 - e_1}{e_2}, \frac{1 - e_2}{e_1}\right)$$

For transfers between non-intersecting orbits, the cost, or characteristic speed, is not very sensitive to the angle $\bar{\omega}_2$:

If the point e_2, P_1/P_2 (Fig. 12) is underneath C_4 the cost of the transfer is:

$$V_{c\infty} = \sqrt{\frac{2\mu}{P_1}} - \sqrt{\frac{\mu A_1}{a_1 P_1}} + \sqrt{\frac{2\mu}{P_2}} - \sqrt{\frac{\mu A_2}{a_2 P_2}} \quad \text{whatever } \bar{\omega}_2 \text{ is.}$$

If the point e_2, P_1/P_2 is above C_4 the cost of the transfer is that of the Hohmann transfer for $\bar{\omega}_2 = 0$:

$$V_{CH} = \sqrt{\frac{\mu P_2}{a_2 A_2}} - \sqrt{\frac{\mu A_1}{a_1 P_1}} + \frac{(A_2 - P_1)\sqrt{2\mu}}{\sqrt{A_2 P_1 (A_2 + P_1)}}$$

For $\bar{\omega}_2 \neq 0$, it is such that:

$$V_{CH} \leq V_C \leq V_{CH}\left(1 + \frac{3(A_2 - P_1)^2}{32 A_2 P_1}\right)$$

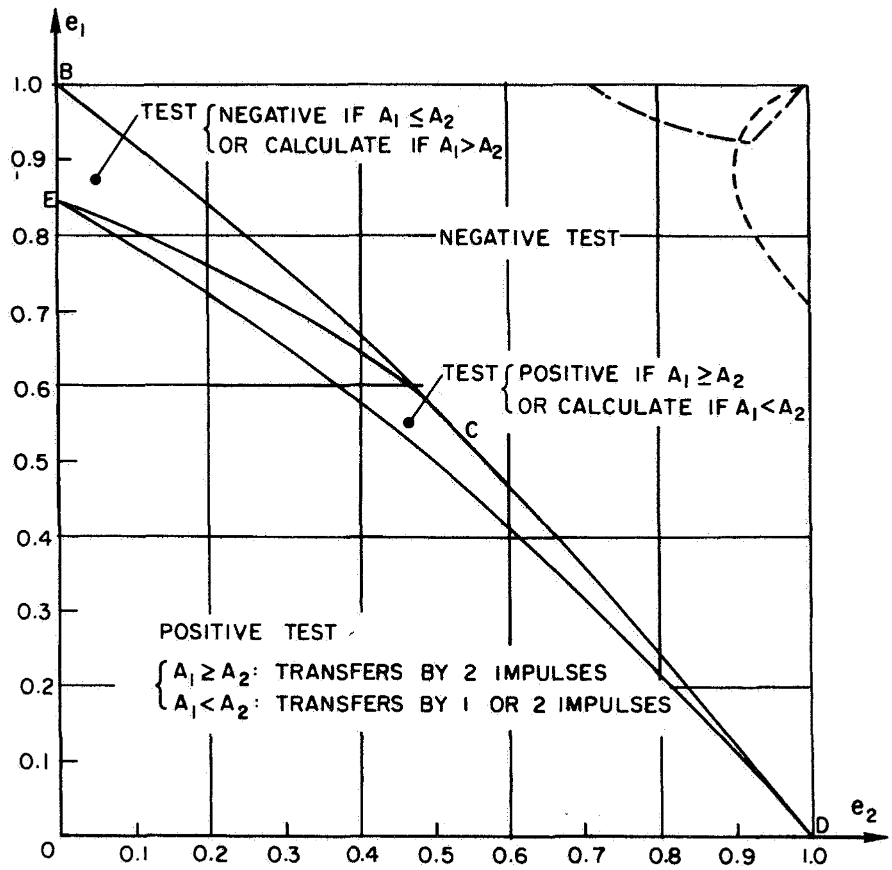
Fig. 13 applies in the case where the given orbits intersect; e_2 is the abscissa, and e_1 the ordinate.

Fig. 13. Transfers between intersecting coplanar orbits $P_1 \leq P_2 < A_1 \lesseqgtr A_2$.

Test: $Z_1\sqrt{A_2} + Z_2\sqrt{A_1} - \sqrt{A_1 + A_2} \gtrless 0$

$$Z(e) = \frac{\sqrt{1+e} - e\sqrt{2}}{\sqrt{1-e}}$$

If $|\bar{\omega}_2| \geq 22°$ the optimal transfer cannot be "by 3 impulses."



This figure helps in applying the test:

$$z_1\sqrt{A_2} + z_2\sqrt{A_1} - \sqrt{A_1+A_2} > 0$$

$$\text{with } z(e) = \frac{\sqrt{1+e} - e\sqrt{2}}{\sqrt{1-e}}$$

which indicates particularly in the case $\bar{\omega}_2 = \pi$:

Negative test: optimal transfer "through infinity."

Positive test: optimal transfer "by 2 impulses" where the transfer ellipse is bi-tangential at the apogees.

One distinguishes 2 cases:

1. $P_1 \leq P_2 \leq A_2 \leq A_1$ (then $e_1 \geq e_2$). The optimal transfer can be one of three types (in departing from the orbit A_1, P_1):

I. By 2 impulses (one accelerating thrust, then a brake).
II. By 3 impulses (an accelerating thrust, then 2 brakes).
III. "Through infinity."

If the test is positive (or zero), the optimal transfer is by 2 impulses.

If the test is negative, the conditions sufficient for "passage through infinity" should be applied: $\frac{8}{9}(\sqrt{2}-1)(1-\cos\bar{\omega}_2) \geq \frac{1-e_1}{e_2}$
or else P_1, P_2, and A_1 satisfying the condition of Fig. 11, can lead to the conclusion of optimality of the transfer "through infinity."

Otherwise, the optimal transfer is either by 2 impulses or "through infinity;" it can even be by 3 impulses if $|\bar{\omega}_2| < 22^\circ$ and if the point (e_2, e_1) is above the mixed dashed line in the upper right of Fig. 13.

2. $P_1 < P_2 < A_1 < A_2$. This case is the most complicated. Any of the following types can be optimal (in departing from the orbit P_1, A_1):

I. With one impulse (accelerating thrust).
II. With 2 accelerating impulses.
III. With 2 impulses (one accelerating thrust, then a brake).
IV. With three impulses.
V. "Through infinity."

If the test is positive or zero, the optimal transfer is by one or two impulses.

If the test is negative, two conditions sufficient for "passage through infinity" are: $(\sqrt{2}-1)(1-\cos\bar{\omega}_2) \geq \min\left(\frac{1-e_1}{e_2}, \frac{1-e_2}{e_1}\right)$
or else P_1, P_2, and A_2 satisfying the condition of Fig. 11. The optimality of the transfer "through infinity" can thus be determined.

As before, otherwise the optimal transfer is either "by one or two impulses" or "through infinity;" it can be "by 3 impulses" if $|\bar{\omega}_2| < 22^\circ$ and if the point (e_2, e_1) is to the right of the dotted line in the upper right of Fig. 13.

Contrary to the case of non-intersecting orbits, the transfers between intersecting orbits have a characteristic speed which is very sensitive to the angle $\bar{\omega}_2$, as one can judge from the following example:

Example: $\mu = 1$, $P_1 = 1$, $P_2 = 2$, $A_1 = 3$, $A_2 = 4$.
$\bar{\omega}_2$ varies from 0° to 180°.
$\bar{\omega}_2 < 60^\circ$--the given orbits are interior.
$\bar{\omega}_2 = 60^\circ$--the given orbits are tangent.
$\bar{\omega}_2 > 60^\circ$--the given orbits intersect.

One obtains the following table:

$\bar{\omega}_2$	0°		60.38°		60.62°		180°
Optimal mode of transfer		Two accelerating impulses		One accelerating impulse		One accelerating impulse, then a braking impulse	
Characteristic speed of transfer	0.1322		0.1519		0.1522		0.2636

Remark: The curves traced on Fig. 13 have the following equations:

$$\left.\begin{array}{ll}\text{Curve BC:} & z_1^2 + z_2^2 = 1 \\ \text{Curve CE:} & z_1 + z_2 = \sqrt{2}\end{array}\right\} \quad z(e) = \frac{\sqrt{1+e} - e\sqrt{2}}{\sqrt{1-e}}$$

$$\text{Curve CD:} \quad \sqrt{2(1+e_1)(1+e_2)} = \sqrt{1 - e_1 e_2} + e_1\sqrt{1+e_2} + e_2\sqrt{1+e_1}$$

These three curves are tangent (slope = -1) at C ($e_1 = e_2 = 0.53533$).

Curve DE: This is a transformation of the curve C_3 (Fig. 12) with:

$$\frac{1 - e_1}{1 + e_1} = \frac{P_1}{P_2}$$

The curves DE and CD are tangent at D (slope = $-\dfrac{\sqrt{2}+1}{2}$).

The curve DE passes through the point $e_1 = e_2 = \frac{1}{2}$.

The limiting curves of the regions of transfer "by 3 impulses" (upper right) are traced "by judgment." The one in mixed, dashed lines passes through $e_1 = e_2 = e_0 \sim 0.925$, and through $e_1 = 1$, $e_2 = 0.7127$. The one in dotted lines passes through $e_1 = e_2 = 1$ (slope = 0.36) and through $e_2 = 1$, $e_1 = 0.7127$.

4. FOURTH PART. THE PRACTICAL POINT OF VIEW

In practice rockets are incapable of realizing theoretical impulses, and also the transfers ought to be effected in a finite time. It is then necessary to adopt a solution in the neighborhood of the optimum. One decomposes each impulse into a limited number of small thrusts which one effects using the maximal thrust of the rocket, on arcs of central angle $\Delta\alpha$, situated in the vicinity of the optimal theoretical impulse point (and obviously at intervals of a revolution to one another). The loss relative to the optimum is then less than:

$$\frac{r}{24p}\,\Delta\alpha^2 \quad \text{(where p is the semi-latus rectum).}$$

On the other hand, it sometimes occurs that the theoretical optimal transfer is a "passage through infinity," which is therefore not physically realizable. One can approach it by utilizing very elongated ellipses which are eventually joined at great distance by a large circle or a large ellipse. If T is the desired transfer time, the loss relative to the optimum is of order $T^{-2/3}$ if the orbits are coaxial and aligned (because the elongated ellipses are then directly joinable), and of order $T^{-1/3}$ if the given orbits are not coaxial and aligned.

Finally, here is an interesting result in the case where the desired transfer time T is not very long (let us choose a few orbital revolutions).

Let us pose $k = \dfrac{\max(A_1,\ A_2)}{\min(P_1,\ P_2)}$ $(k \geq 1)$

To begin with, for unlimited time T:

If $k < 21$: the optimal transfer is never by 3 impulses.

If $k < 3.3041$: the optimal transfer is by one or two impulses (which one can see with the help of Fig. 12 and the test of Fig. 13).

If $k < 8.792$ and if the given orbits are interior, the optimal transfer is by 2 impulses (which one can see in Fig. 12).

Therefore, in particular for $3.3041 < k < 15.582$ the optimal transfer can be by one or two impulses or else "through infinity."

Let us suppose that we have the rather frequent last case, $3.3041 < k < 15.582$, and that the optimal transfer is "through infinity." There is, among the solutions with one or 2 impulses, an optimal transfer which we will call the "close optimal transfer" and which has, for this case, a characteristic speed, which is greater than that for the transfer "through infinity."

Then let j be such that: $$\frac{1}{\sqrt{2k}} + \frac{k-1}{\sqrt{k(k+1)}} = \frac{1}{\sqrt{2j}} + \frac{j-1}{\sqrt{j(j+1)}}$$

with $3.3041 < k < 15.582 < j$

(j decreases from $+\infty$ to 15.582 when k increases from 3.3041 to 15.582.) The announced result is: the "close optimal transfer" is better (from the point of view of characteristic speed) than every transfer which does not involve any impulse or thrust (finite or infinitely small) beyond the distance $j \cdot \min(P_1,\ P_2)$: Therefore if j is sufficiently high, it is evident that the "close optimal transfer" is optimal for T of the order of several orbital revolutions.

Here are several values of j for values of k:

k		j	
3.3041	(= root of $8k^3 - 33k^2 + 22k = 1$)	$+\infty$	about
3.5	....	7630	about
4	....	713	about
5	....	160	about
7	....	53	about
11.5	....	22.1	about
15.582	....	15.582 = root of: $j^3 = 15j^2 + 9j + 1$.	

The "close optimal" transfer is the Hohmann transfer if the given orbits are aligned and coaxial ($\bar{\omega}_2 = 0$). It is a transfer by a bi-tangential ellipse at the two apogees or at the two perigees if the given orbits are opposing and coaxial ($\bar{\omega}_2 = 180^\circ$). In every case it is locally optimal and fulfills the conditions on the useful angle, of commutation, and of sense of direction of the commutations defined in Part 2.

These results cannot be extended to the case of transfers between non-coplanar orbits.

Remark: If the two given coplanar orbits are non-intersecting, the relation (only valuable if $V_{CH} \leq V_{C\infty}$):

$$V_{CH} \leq V_C \leq V_{CH}\left(1 + \frac{3(k-1)^2}{32k}\right)$$

is always valuable if one takes V_C = characteristic speed of the close optimal transfer (V_{CH} designating the characteristic speed of the corresponding Hohmann transfer obtained by taking $\bar{\omega}_2 = 0$). This relation permits the attainment of a value approaching the necessary characteristic speed.

CONCLUSION

The optimal transfers, from the point of view of characteristic speed, between coplanar elliptic orbits (time open) are generally realizable either by 2 impulse transfers, with an intermediate ellipse if the given orbits are slightly different, or by transfers (through infinity" if the given orbits are very different. These transfers are realizable in practice, with as small a loss as one wishes, by interposing very elongated ellipses joined at a great distance by a large circle or a large ellipse. There is nevertheless a small proportion of optimal transfer cases with one impulse or with 3 impulses. These latter, which have 2 intermediary ellipses, only appear if the sum of the eccentricities of the initial and final orbits is greater than 1.712.

The utilized impulses are always within one of the acute angles formed by the tangent and the local horizontal and are always less than 26.2° from the local horizontal. Of course, these impulses can be fractioned as much as one wishes if the thruster cannot furnish them at one time. It then effects each fraction at an interval of one revolution.

When the optimal transfer is not "through infinity" the research of the optimal impulses and of the intermediate ellipse(s) is in general very complicated, but it is easy in the following cases: transfers between: aligned coaxial orbits (perigees on the same side); opposing coaxial orbits (perigees on opposite sides); equal orbits; orbits of which one is of eccentricity close to zero or one.

It will certainly be of great interest to research the influence of a time limit and the loss which results therein. This is true as much in the cases of transfers "through infinity" as for those where a weak thruster obliges the use of thrust arcs.

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